§ 14.4

In Exercises 1–6, **(a)** express dw/dt as a function of t, both by using the Chain Rule and by expressing w in terms of t and differentiating directly with respect to t. Then **(b)** evaluate dw/dt at the given value of t.

5. $w = 2ye^x - \ln z, \quad x = \ln(t^2+1), \quad y = \tan^{-1} t, \quad z = e^t;$
$t = 1$

(a).
$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial w}{\partial z}\cdot\frac{dz}{dt}$$

$$= (2ye^x)\cdot\left(\frac{2t}{t^2+1}\right) + (2e^x)\left(\frac{1}{t^2+1}\right) + \left(-\frac{1}{z}\right)\cdot(e^t)$$

$$= \frac{4(\tan^{-1}t)\,e^{\ln(t^2+1)}\cdot t}{t^2+1} + \frac{2e^{\ln(t^2+1)}}{t^2+1} - \frac{e^t}{e^t}$$

$$= 4t\tan^{-1}(t) + 1$$

$$w = 2ye^x - \ln z$$

$$= 2\cdot\tan^{-1}t\cdot e^{\ln(t^2+1)} - \ln(e^t)$$

$$= 2\tan^{-1}(t)\cdot(t^2+1) - t.$$

$$\frac{dw}{dt} = 2\left(\frac{1}{t^2+1}\right)\cdot(t^2+1) + 2\tan^{-1}(t)(2t) - 1$$

$$= 1 + 4t\tan^{-1}(t)$$

(b).
$$\left.\frac{dw}{dt}\right|_{t=1} = 1 + 4\cdot 1\cdot\frac{\pi}{4} = 1+\pi$$

In Exercises 11 and 12, **(a)** express $\partial u/\partial x$, $\partial u/\partial y$, and $\partial u/\partial z$ as functions of x, y, and z both by using the Chain Rule and by expressing u directly in terms of x, y, and z before differentiating. Then **(b)** evaluate $\partial u/\partial x$, $\partial u/\partial y$, and $\partial u/\partial z$ at the given point (x, y, z).

12. $u = e^{qr}\sin^{-1} p, \quad p = \sin x, \quad q = z^2 \ln y, \quad r = 1/z;$
$(x, y, z) = (\pi/4, 1/2, -1/2)$

(a) $$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial x} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x}$$

$$= e^{qr}\cdot\frac{1}{\sqrt{1-p^2}}\cdot\cos x + 0 + 0$$

$$= e^{z\ln y}\cdot\frac{1}{\sqrt{1-\sin^2 x}}\cos x = \begin{cases} y^z & \text{if } \cos x > 0 \\ -y^z & \text{if } \cos x < 0 \end{cases}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial y} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial y} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y}$$

$$= 0 + re^{qr}\sin^{-1}p\cdot\frac{z^2}{y} + 0.$$

$$= \frac{1}{z}y^z\sin^{-1}(\sin x)\cdot\frac{z^2}{y}$$

$$= x\,y^{z-1}z \quad \text{if } -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

(Note $\sin^{-1}(\sin x) \ne x$ for $x < -\frac{\pi}{2}$ or $x > \frac{\pi}{2}$).

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial z} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial z} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z}$$

$$= 0 + re^{qr}\sin^{-1}(p)\cdot 2z\ln y + qe^{qr}\sin^{-1}(p)\cdot\left(-\frac{1}{z^2}\right)$$

$$= \frac{1}{z}y^z\sin^{-1}(\sin x)\cdot 2z\ln y + z^2\ln y\, y^z\sin^{-1}(\sin x)\cdot\left(-\frac{1}{z^2}\right)$$

$$= 2y^z\ln y\, x - \ln y\, y^z x = x\,y^z\ln y \quad \text{if } -\frac{\pi}{2} \le x \le \frac{\pi}{2}.$$

$$u = e^{zr}\sin^{-1}p$$

$$= e^{z\ln y}\cdot\sin^{-1}(\sin x)$$

$$= y^z x \quad \text{if} \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}.$$

Then For $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$,

$$\frac{\partial u}{\partial x} = y^z, \quad \frac{\partial u}{\partial y} = z\,y^{z-1}x, \quad \frac{\partial u}{\partial z} = x\,y^z\ln y$$

(b). At $(x, y, z) = \left(\frac{\pi}{4}, \frac{1}{2}, -\frac{1}{2}\right)$,

$$\frac{\partial u}{\partial x} = \sqrt{2}, \quad \frac{\partial u}{\partial y} = -\frac{\pi}{4}\sqrt{2}, \quad \frac{\partial u}{\partial z} = -\frac{\pi}{4}\sqrt{2}\ln 2$$

52. $F(x) = \int_{x^2}^{1} \sqrt{t^3 + x^2}\, dt$

Denote $G(u, x) = -\int_1^u \sqrt{t^3 + x^2}\, dt$.

Then $F(x) = G(x^2, x)$.

Fundamental Thm of Calculus

$\Rightarrow \frac{\partial G}{\partial u} = -\sqrt{u^3 + x^2}$

$$\frac{dG}{dx}\Big|_{u=x^2} = \frac{\partial G}{\partial u}\Big|_{u=x^2} \cdot \frac{\partial u}{\partial x}\Big|_{u=x^2} + \frac{\partial G}{\partial x}\Big|_{u=x^2}$$

$$= -\sqrt{x^6 + x^2} \cdot 2x - \int_1^{x^2} \frac{2x}{2\sqrt{t^3 + x^2}}\, dt$$

$$= -2x\sqrt{x^6 + x^2} - \int_1^{x^2} \frac{x}{\sqrt{t^3 + x^2}}\, dt.$$

§14.6

In Exercises 1–8, find equations for the

(a) tangent plane and

(b) normal line at the point P_0 on the given surface.

2. $x^2 + y^2 - z^2 = 18, \quad P_0(3, 5, -4)$

(a). $f(x,y,z) := x^2 + y^2 - z^2$

$\nabla f = 2x\vec{i} + 2y\vec{j} - 2z\vec{k}.$

$\nabla f(3,5,-4) = 6\vec{i} + 10\vec{j} + 8\vec{k}.$

$\therefore$ Tangent plane :

$\nabla \vec{f} \cdot ((x,y,z) - (3,5,-4)) = 0.$

$\Leftrightarrow 6(x-3) + 10(y-5) + 8(z+4) = 0$

$\Leftrightarrow 3x + 5y + 4z = 18.$

(b). Tangent line :

$(3,5,-4) + t(6,10,8).$

In Exercises 9–12, find an equation for the plane that is tangent to the given surface at the given point.

9. $z = \ln(x^2 + y^2)$, $(1, 0, 0)$

$$f(x, y, z) := \ln(x^2 + y^2) - z$$

$$\nabla f = \frac{2x}{x^2 + y^2}\vec{i} + \frac{2y}{x^2 + y^2}\vec{j} - \vec{k}$$

$$\nabla f(1, 0, 0) = 2\vec{i} - \vec{k}.$$

tangent plane :

$$\nabla f \cdot ((x, y, z) - (1, 0, 0)) = 0$$

$$\Leftrightarrow \quad 2(x - 1) - z = 0.$$

$$\Leftrightarrow \quad 2x - z - 2 = 0.$$

19. By about how much will

$$f(x, y, z) = \ln\sqrt{x^2 + y^2 + z^2}$$

change if the point $P(x, y, z)$ moves from $P_0(3, 4, 12)$ a distance of $ds = 0.1$ unit in the direction of $3\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$?

$$\vec{u} := \frac{3\vec{i} + 6\vec{j} - 2\vec{k}}{\| 3\vec{i} + 6\vec{j} - 2\vec{k} \|} = \frac{3}{7}\vec{i} + \frac{6}{7}\vec{j} - \frac{2}{7}\vec{k}.$$

$$f(x, y, z) = \ln\sqrt{x^2 + y^2 + z^2}$$

$$= \frac{1}{2}\ln(x^2 + y^2 + z^2)$$

$$\nabla f = \frac{x}{x^2 + y^2 + z^2}\vec{i} + \frac{y}{x^2 + y^2 + z^2}\vec{j} + \frac{z}{x^2 + y^2 + z^2}\vec{k}.$$

$$D_{\vec{u}} f(3, 4, 12) = \frac{3}{7} \cdot \frac{3}{169} + \frac{6}{7} \cdot \frac{4}{169} + \left(-\frac{2}{7}\right) \cdot \frac{12}{169}$$

$$= \frac{9}{7 \cdot 169} = \frac{9}{1183}.$$

$$df \approx D_{\vec{u}} f \cdot ds$$

$$= \frac{9}{1183} \cdot 0.1$$

$$= \frac{9}{11830}$$

$$\approx 7.61 \times 10^{-4}.$$

In Exercises 25–30, find the linearization $L(x, y)$ of the function at each point.

29. $f(x, y) = e^x \cos y$ at **a.** $(0, 0)$, **b.** $(0, \pi/2)$

$$\nabla f = e^x \cos y \, \vec{i} - e^x \sin y \, \vec{j}$$

(a). $\nabla f(0, 0) = \vec{i}$

$f(0, 0) = 1$

$L(x, y) = 1 + x$

(b). $\nabla f(0, \pi/2) = -\vec{j}$

$f(0, \pi/2) = 0.$

$L(x, y) = -(y - \pi/2) = \frac{\pi}{2} - y.$

Find the linearizations $L(x, y, z)$ of the functions in Exercises 39–44 at the given points.

39. $f(x, y, z) = xy + yz + xz$ at

a. $(1, 1, 1)$ **b.** $(1, 0, 0)$ **c.** $(0, 0, 0)$

$$\nabla f = (y+z)\vec{i} + (x+z)\vec{j} + (x+y)\vec{k}.$$

(a). $\nabla f(1,1,1) = 2\vec{i} + 2\vec{j} + 2\vec{k}.$

$f(1,1,1) = 3$

$L(x,y,z) = 3 + 2(x-1) + 2(y-1) + 2(z-1)$

$= 2x + 2y + 2z - 3.$

(b). $\nabla f(1,0,0) = \vec{j} + \vec{k}.$

$f(1,0,0) = 0$

$L(x,y,z) = 0 + (y-0) + (z-0)$

$= y + z.$

(c). $\nabla f(0,0,0) = 0$

$f(0,0,0) = 0$

$L(x,y,z) = 0.$

§ 14.7

45. Show that (0, 0) is a critical point of $f(x, y) = x^2 + kxy + y^2$ no matter what value the constant k has. (*Hint:* Consider two cases: $k = 0$ and $k \neq 0$.)

$f_x(x,y) = 2x + ky \qquad f_x(0,0) = 0$

$f_y(x,y) = 2y + kx \qquad f_y(0,0) = 0.$

$\therefore$ $(0,0)$ is a critical point of f.

Or find all critical points of f.

(Case 1): Suppose $k = 0$.

Then $\nabla f = (2x, 2y)$.

$(0, 0)$ is the only critical point of f.

(Case 2): Suppose $k \neq 0$.

Then $\nabla f = (2x + ky, 2y + kx)$.

$2x + ky = 0 \iff x = -\frac{k}{2} y.$

$2y + kx = 0 \iff 2y - \frac{k^2}{2} y = 0$

$\iff y = 0$ or $k^2 - 4 = 0$

If $y = 0$, then $x = -\frac{k}{2} y = 0$.

If $k^2 - 4 = 0$, then $k = \pm 2$,

$x = -\frac{k}{2} y = \mp y.$

$\therefore$ In any cases, $(x, y) = (0, 0)$ is a critical point.

47. If $f_x(a, b) = f_y(a, b) = 0$, must f have a local maximum or minimum value at (a, b)? Give reasons for your answer.

Without lost of generality,

assume $a = b = 0$.

(In general, we can shift (a,b) to $(0,0)$ by defining $g(x,y) = f(x+a, y+b)$ then $g(0,0) = f(a,b)$.)

Choose $f(x,y) = xy$

then $f_x(x,y) = y \quad f_y(x,y) = x$

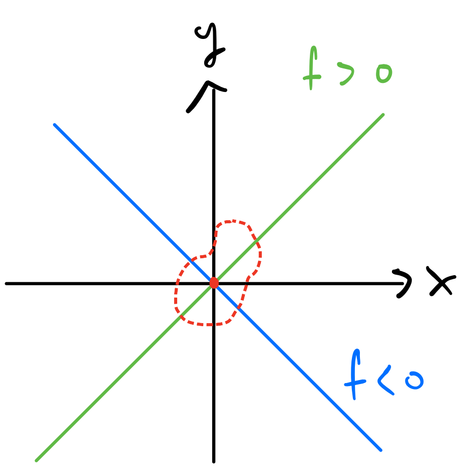
$f_x(0,0) = 0 \quad f_y(0,0) = 0.$

f satisfy the assumption of the question.

$f(0,0) = 0.$

$f(t,t) = t^2 > 0 = f(0,0) \quad \forall t \neq 0$

$f(t,-t) = -t^2 < 0 = f(0,0) \quad \forall t \neq 0.$



For any open neighbourhood of $(0,0)$,

$\exists$ some pt s.t. $f < 0$ and $f > 0$

$\therefore f(0,0)$ is neither local max. nor local min.

Or : $f_{xx} = 0, \quad f_{yy} = 0$

$f_{xy} = 1$

$\therefore \quad f_{xx} f_{yy} - f_{xy}^2 = -1 < 0$

$\therefore$ $(0, 0)$ is a saddle point of f.

$\therefore$ Not local max and

Not local min.